Date

Solving system of linear equations

Consider the 3×3 system of linear equations

$$\begin{cases} x + 2y + z = 4 & \text{------- (i)} \\ 3x + 8y + 7z = 20 & \text{------- (ii)} \\ 2x + 7y + 9z = 23 & \text{------- (iii)} \end{cases}$$

Each eqn. actually represents a plane in space.
Hence, the solution may consist of

(i) one solution ⟵ planes intersect at 1 pt.

(ii) infinitely many solutions ⟵ planes intersect along a line

(iii) no solution at all ⟵ do not intersect at all.

To solve the system we construct the <u>Augmented Matrix</u> of the system as

$$\underbrace{\left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 3 & 8 & 7 & 20 \\ 2 & 7 & 9 & 23 \end{array}\right]}$$

matrix of the coefficients on the left

The constant column on the right.

Then we employ row operations by

(i) switching between any 2 rows

(ii) multiply a row by a non-zero constant

(iii) add a non-zero constant multiple of a row to another row.

Final Goal: Through row operations we reduce the Augmented matrix into an upper triangular matrix

diagonal of the coeff. matrix

end up with a zero lower triangular block

i.e. every entry under the diagonal of the coefficient matrix is zero. Back to

$$\text{Eliminate using}\left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ \cancel{3} & 8 & 7 & 20 \\ \cancel{2} & 7 & 9 & 23 \end{array}\right] \xrightarrow[-2R_1+R_3]{-3R_1+R_2} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & 2 & 4 & 8 \\ 0 & \cancel{3} & 7 & 15 \end{array}\right]$$

now eliminate 3 using R_2 as the pivot row.

R_1 as the pivot row (the row that doesn't change during the operation).

$$\xrightarrow{\frac{R_2}{2} \to R_2} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & 1 & 2 & 4 \\ 0 & \cancel{3} & 7 & 15 \end{array}\right] \xrightarrow{-3R_2+R_3} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

Now, Solve the system by backward subsitution, the original system is transformed into an equivalent one

$$x + 2y + z = 4 \quad \text{---- (i)}$$
$$y + 2z = 4 \quad \text{---- (ii)}$$
$$z = 3 \quad \text{---- (iii)}$$

$\boxed{z=3}$, subsituting into (ii) $y + 6 = 4 \Rightarrow \boxed{y=-2}$

subsituting all these back to (i), $x - 4 + 3 = 4$

$\Rightarrow \boxed{x=5}$ Planes intersect at one pt.

Hence, $x = 5$, $y = -2$, $z = 3$ is the solution.

Ex. Solving
$$\begin{cases} 3x - 8y + 10z = 22 \\ x - 3y + 2z = 5 \\ 2x - 9y - 8z = -11 \end{cases}$$

$$\left[\begin{array}{ccc|c} 3 & -8 & 10 & 22 \\ 1 & -3 & 2 & 5 \\ 2 & -9 & -8 & -11 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 1 & -3 & 2 & 5 \\ 3 & -8 & 10 & 22 \\ 2 & -9 & -8 & -11 \end{array}\right]$$

$$\xrightarrow[-2R_1 + R_3]{-3R_1 + R_2} \left[\begin{array}{ccc|c} 1 & -3 & 2 & 5 \\ 0 & 1 & 4 & 7 \\ 0 & -3 & -12 & -21 \end{array}\right] \xrightarrow{-3R_2 + R_3} \left[\begin{array}{ccc|c} 1 & -3 & 2 & 5 \\ 0 & 1 & 4 & 7 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Thus, we have
$$\begin{cases} x - 3y + 2z = 5 \text{ ---- (i)} \\ y + 4z = 7 \text{ ---- (ii)} \end{cases}$$

picking z to be our free variable i.e.

set $z = t$, $-\infty < t < \infty$

(ii) $\Rightarrow$ $y = 7 - 4t$

Substituting $z = t$, $y = 7 - 4t$ back to (i)

$x - 3(7-4t) + 2t = 5$

$\Rightarrow x - 21 + 12t + 2t = 5$

$\Rightarrow x = 26 - 14t$

Hence, we have $x = 26 - 14t$, $y = 7 - 4t$, $z = t$, $-\infty < t < \infty$

$\therefore$ The 3 planes intersect along a straight line //.

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Revisit the example of finding the line of intersection between the lines
$$\begin{cases} 2x+3y-4z=-7 \\ 3x-2y+3z=6 \end{cases}$$

we could simply solve the system by row reduction.

$$\left[\begin{array}{ccc|c} 2 & 3 & -4 & -7 \\ 3 & -2 & 3 & 6 \end{array}\right] \xrightarrow{\frac{R_1}{2}\to R_1} \left[\begin{array}{ccc|c} 1 & 3/2 & -2 & -7/2 \\ 3 & -2 & 3 & 6 \end{array}\right]$$

$$\xrightarrow{-3R_1+R_2} \left[\begin{array}{ccc|c} 1 & 3/2 & -2 & -7/2 \\ 0 & -\frac{13}{2} & 9 & \frac{33}{2} \end{array}\right]$$

Thus we have,
$$\begin{cases} x+\frac{3}{2}y-2z=-7/2 & \text{------ (i)} \\ -\frac{13}{2}y+9z=\frac{33}{2} & \text{------ (ii)} \end{cases}$$

Set $\boxed{z=s}$, $-\infty < s < \infty$ taking z as a "free variable", s is known as a parameter which could take any values.

(ii) $\Rightarrow$ $\frac{13y}{2} = 9s - 33/2 \Rightarrow \boxed{y = \frac{18}{13}s - \frac{33}{13}}$

Substituting all these into (i) $\Rightarrow$

$$x+\frac{3}{2}\left(\frac{18}{13}s-\frac{33}{13}\right)-2s=-7/2$$

$$\Rightarrow x+\frac{27}{13}s-\frac{99}{26}-2s=-7/2$$

$$\Rightarrow x+\frac{1}{13}s=\frac{99}{26}-7/2=\frac{8}{26}=\frac{4}{13}$$

$$\Rightarrow \boxed{x=-\frac{s}{13}+\frac{4}{13}}$$

Thus $x = \frac{-5}{13} + \frac{4}{13}$, $y = \frac{18}{13}s - \frac{33}{13}$, $z = s \quad -\infty < s < \infty$

is the line of intersection.

Recall our old solution : $x = t + 3/13$, $y = -18t - \frac{15}{13}$

(the example on page 27-28) $z = -13t + 1 \quad -\infty < t < \infty$

To see they are equivalent, set x co-ordinate equal,

$$t + 3/13 = \frac{-5}{13} + \frac{4}{13} \Rightarrow t = \frac{-5}{13} + \frac{1}{13}$$

Substituting into $y = -18t - \frac{15}{13} = -18\left(\frac{-5}{13} + \frac{1}{13}\right) - \frac{15}{13}$

$$= \frac{18s}{13} - \frac{18}{13} - \frac{15}{13}$$

$$= \frac{18}{13}s - \frac{33}{13} //.$$

In the same way, $z = -13\left(\frac{-5}{13} + \frac{1}{13}\right) + 1$

$$= s //.$$

Hence, the two parametric representations are equivalent to each other //.

IV. Motion in Space

Vector-valued functions

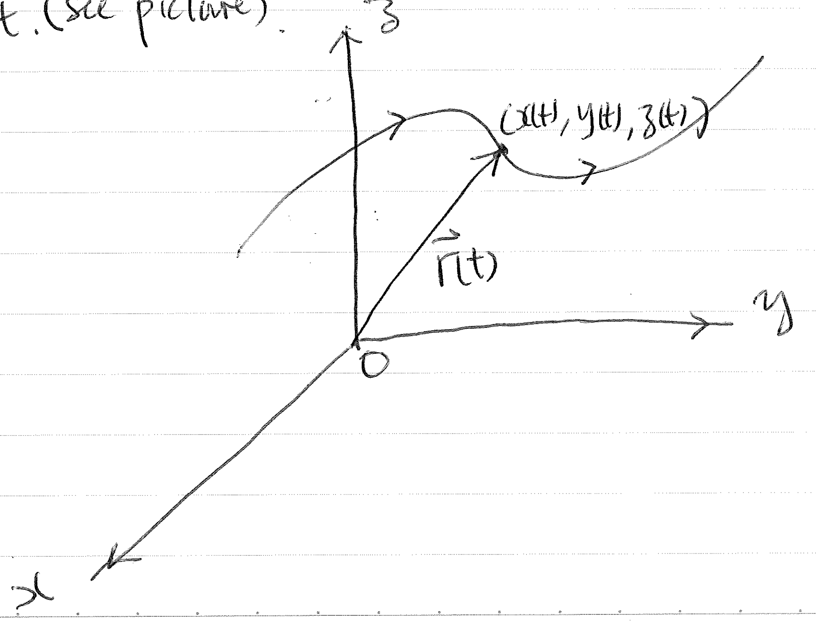
Consider a mapping $\vec{r} \mid [a,b] \longrightarrow V_3$ which takes the form $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$ $t \in [a,b]$.

Thus, corresponding to any $t \in [a,b]$, there is a vector
$\vec{r}(t) = \langle x(t), y(t), z(t) \rangle \in V_3$
Where $x(t), y(t), z(t)$ are respectively functions of t

<u>Remark</u>: We could have replaced $[a,b]$ with any interval I such as $I = (-\infty, \infty)$.

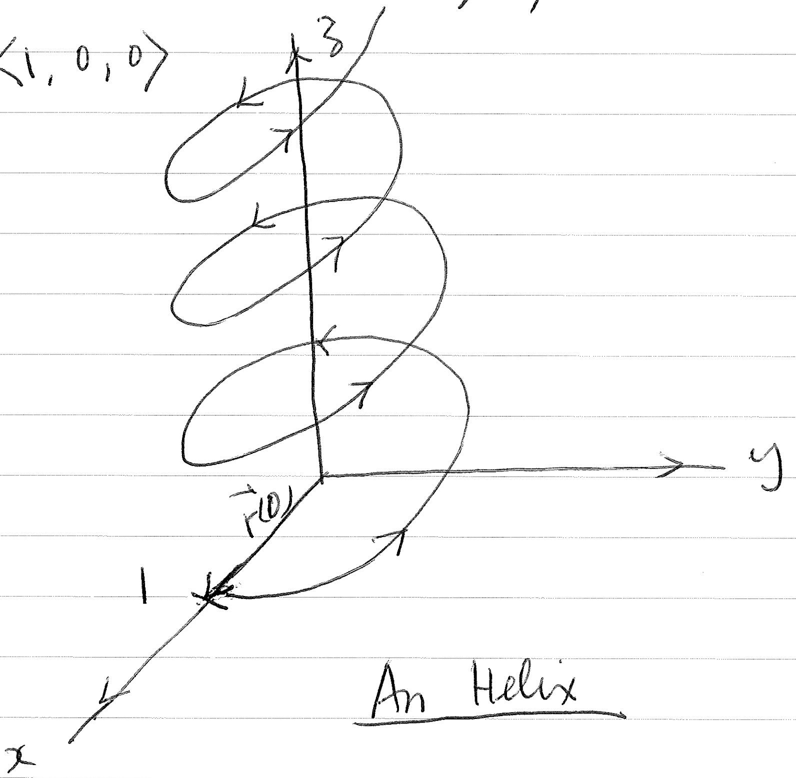
Geometric Interpretation

The most natural way is to take t to be the time variable, and $x(t), y(t), z(t)$ to the x, y & z co-ordinates of an moving object in space. As t changes, the position $(x(t), y(t), z(t))$ of the object changes and trace out a trajectory or curve in space. In this case, $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$ becomes the position vector pointing from the origin of the co-ordinate system to the object at time t. (see picture)



Ex. $\vec{r}(t) = \langle \cos t, \sin t, t \rangle$, $t \in [0, \infty)$

$\vec{r}(0) = \langle 1, 0, 0 \rangle$



An Helix

Differentiation of vector-valued function

Definition:

$$\vec{r}'(t) = \lim_{h\to 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h}$$

$$= \lim_{h\to 0} \frac{\langle x(t+h), y(t+h), z(t+h)\rangle - \langle x(t), y(t), z(t)\rangle}{h}$$

$$= \lim_{h\to 0} \left\langle \frac{x(t+h) - x(t)}{h}, \frac{y(t+h)-y(t)}{h}, \frac{z(t+h)-z(t)}{h} \right\rangle$$

$$= \left\langle \lim_{h\to 0} \frac{x(t+h)-x(t)}{h}, \lim_{h\to 0} \frac{y(t+h)-y(t)}{h}, \lim_{h\to 0} \frac{z(t+h)-z(t)}{h} \right\rangle$$

$= \langle x'(t), y'(t), z'(t) \rangle$ i.e. Differentiate Component-wise

Ex. Back to $\vec{r}(t) = \langle \cos t, \sin t, t \rangle$, $\frac{d\vec{r}(t)}{dt}$ or $\vec{r}'(t) = \langle -\sin t, \cos t, 1 \rangle$ //

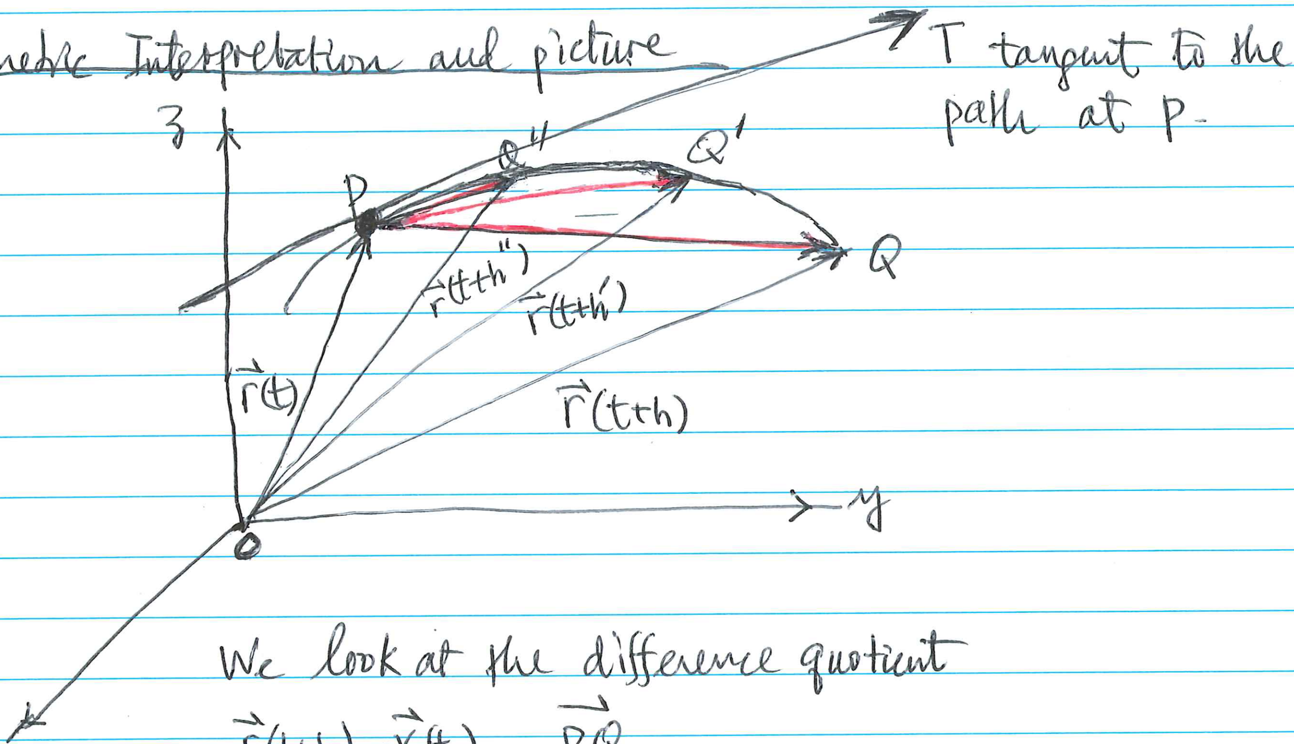
i.e. differentiating componentwise

Defn. Given $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$ to be the position vector of an moving object along a trajectory in space, we define the derivative

$$\vec{r}\,'(t) = \lim_{h \to 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h}$$

as the instantaneous velocity of the object at time t, we denote it by $\vec{v}(t)$.

Geometric Interpretation and picture



We look at the difference quotient

$$\frac{\vec{r}(t+h) - \vec{r}(t)}{h} = \frac{\vec{PQ}}{h}$$

$$\left| \frac{\vec{r}(t+h) - \vec{r}(t)}{h} \right| = \frac{|\vec{PQ}|}{h} \approx \frac{\text{arc}\,PQ}{h}$$

This is the average speed of the moving object from P to Q along the trajectory. It measures how fast along the path the object is moving regardless of its direction of motion.

As $h \to h' \to h'' \to 0$ $|\vec{PQ}|$, $|\vec{PQ'}|$ and $|\vec{PQ''}|$ are getting closer and closer to arcPQ, arcPQ', arcPQ'' respectively. In the limiting process, as $h \to 0$ $\left| \frac{\vec{r}(t+h) - \vec{r}(t)}{h} \right| \longrightarrow$ instantaneous speed of the object

i.e. it measures how fast the object is moving at time t regardless of its direction of motion.

As far as the direction of $\frac{\Delta \vec{r}(t)}{h}$ is concerned, its direction is determine by $\overrightarrow{PQ}$. As $h \to h' \to h'' \to 0$, the directions of $\overrightarrow{PQ}$, $\overrightarrow{PQ'}$ and $\overrightarrow{PQ''}$ would rotates into the direction of T (tagent to the path at P).

Thus, $|\vec{r}'(t)|$ is the instantaneous speed of the object while the direction of $\vec{r}'(t)$ is the same as the direction of motion at time t which is tangential to the path of motion of the object.

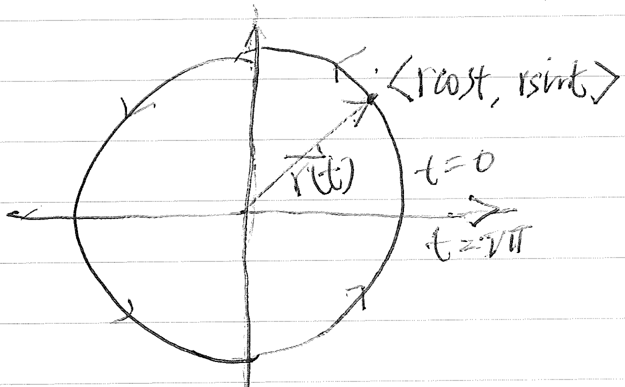
From now on, we set

$$\vec{v}(t) = \lim_{h \to 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h}$$

which is the instantaneous velocity of the moving obj or simply the velocity of the moving object at time t, $s(t) = |\vec{v}(t)|$ is the speed function at time t.

Ex. Consider the circular motion in the plane which is given by

$$\vec{r}(t) = r\langle \cos t, \sin t \rangle \qquad 0 \le t \le 2\pi$$



$$\vec{v}(t) = \vec{r}'(t) = \langle -r\sin t, r\cos t \rangle$$

$$s(t) = |\vec{v}(t)| = \sqrt{r^2\sin^2 t + r^2\cos^2 t} = r$$

Circumference of the circle = distance travelled from $t=0$ to $t=2\pi$

$$= \int_0^{2\pi} s(t)\,dt = \int_0^{2\pi} r\,dt = \Big[rt\Big]_0^{2\pi} = 2\pi r \quad //$$

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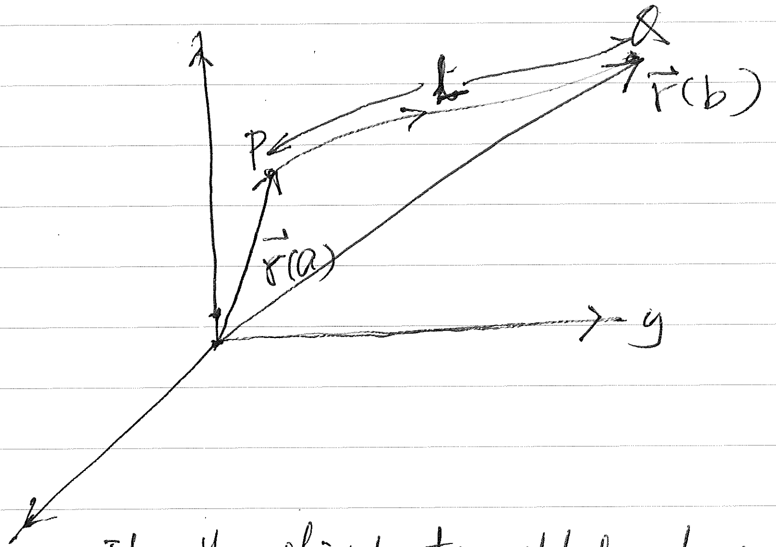
Remarks:

(i) Consider an moving object travelling along a space curve whose vector equation is given by

$$\vec{r}(t) = \langle x(t), y(t), z(t) \rangle \quad a \le t \le b$$



If the object travelled along the curve from P to Q strictly (i.e. there is no reverse of motion during the course). Then, the distance travelled by the object or the length of the curve from P to Q is given by

$$L = \int_a^b s(t)\,dt = \int_a^b |\vec{v}(t)|\,dt$$

(ii) Analogously, as $\vec{v}(t) = \frac{d\vec{r}(t)}{dt}$ or $\vec{r}\,'(t)$, we have

$$\frac{d\vec{v}(t)}{dt} = \vec{v}\,'(t) = \vec{r}\,''(t) \text{ or } \frac{d^2\vec{r}(t)}{dt^2}$$

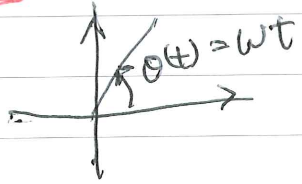
is the acceleration function of the moving object and we denote it by $\vec{a}(t)$.

Ex Uniform circular motion

An object undergoing circular motion in the plane with uniform angular velocity ω has vector equation

$$\vec{r}(t) = \langle r\cos\omega t, r\sin\omega t\rangle$$

(note that $\omega > 0$ implies the direction of rotation is anti-clockwise and is clockwise otherwise)

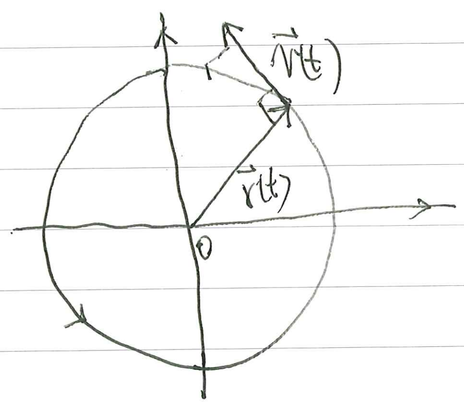


Now that

$$\vec{v}(t) = \langle -r\omega\sin\omega t, r\omega\cos\omega t\rangle$$

We observe that

(i) $\vec{v}(t)\cdot\vec{r}(t) = 0 \Rightarrow \vec{v}(t) \perp \vec{r}(t)$ which is tangential to the trajectory or path of motion as expected



(ii) The speed of the object rotating around the circle is given by $s(t) = |\vec{v}(t)| = \sqrt{r^2\omega^2\sin^2\omega t + r^2\omega^2\cos^2\omega t} = r|\omega|$

(iii) Finally, the acceleration function is given by

$$\vec{a}(t) = \vec{v}'(t) = \langle -r\omega^2\cos\omega t, -r\omega^2\sin\omega t\rangle$$
$$= -\omega^2\langle r\cos\omega t, r\sin\omega t\rangle = -\omega^2\vec{r}(t)$$

The direction of the acceleration is towards the center of a circle with magnitude $|\vec{a}(t)| = \omega^2 r$

This in Physics is known as the centripetal acceleration of the circular motion.

Integration of vector valued functions

Since differentiation of vector-valued function is being handled componentwise, integration being the reverse process of differentiation must be also handled component-wise.

Given $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$

$$\int \vec{r}(t)dt = \left\langle \int f(t)dt, \int g(t)dt, \int h(t)\,dt \right\rangle$$

Ex $\vec{r}(t) = \langle \cos t, \sin t, t \rangle$

$$\int \vec{r}(t)\,dt = \left\langle \int \cos t\,dt, \int \sin t\,dt, \int t\,dt \right\rangle$$

$$= \left\langle \sin t + C_1, -\cos t + C_2, \frac{t^2}{2} + C_3 \right\rangle$$

or we could rewrite it as

$$\left\langle \sin t, -\cos t, \frac{t^2}{2} \right\rangle + \vec{C}$$

where $\vec{C} = \langle C_1, C_2, C_3 \rangle$ is an arbitrary vector constant

Thus, in view of $\vec{v}(t) = \frac{d\vec{r}(t)}{dt}$, $\vec{a}(t) = \frac{d\vec{v}(t)}{dt}$

we could re-express them as,

$$\vec{r}(t) = \int \vec{v}(t)\,dt + \vec{C}, \quad \vec{v}(t) = \int \vec{a}(t)dt + \vec{C}$$

where $\vec{C}$ denotes any arbitrary vector constant.

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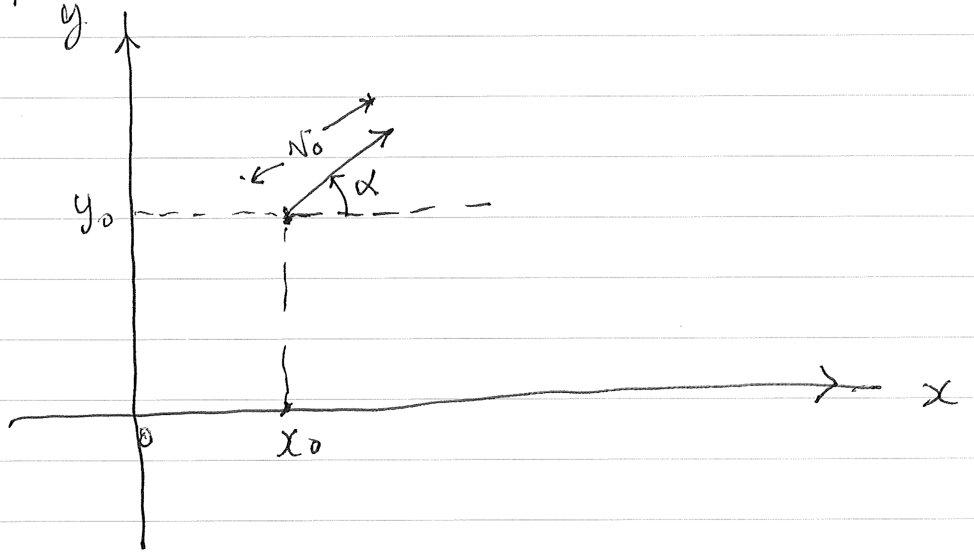
Ex. Projectile motion along a vertical plane.

We shall let the y-axis be the vertical axis and the x-axis be the horizontal axis. Suppose a projectile is being launched from its initial position at (x_0, y_0) with initial speed v_0 making an angle α with the positive x axis.



Predict the trajectory of the subsequent motion of the projectile.

<u>Solution</u>

Assuming the projectile is only under gravitational influence, we have

$$\vec{a}(t) = \langle 0, -g \rangle \quad \text{when } g \text{ is the gravitational acceleration}$$

$$\Rightarrow \vec{v}(t) = \int \vec{a}(t)\,dt + \vec{C}$$

$$= \left\langle \int 0\,dt, \int -g\,dt \right\rangle + \langle C_1, C_2 \rangle$$

$$= \langle C_1, -gt + C_2 \rangle$$

But $\vec{v}(0) = \langle C_1, C_2 \rangle = \langle v_0 \cos\alpha, v_0 \sin\alpha \rangle$

$$\Rightarrow C_1 = v_0 \cos\alpha, \quad C_2 = v_0 \sin\alpha$$

$$\Rightarrow \vec{v}(t) = \langle v_0 \cos\alpha, -gt + v_0 \sin\alpha \rangle$$

Similarly,

$$\vec{r}(t) = \int \vec{v}(t)\,dt + \vec{C}$$

$$= \left\langle \int v_0\cos\alpha\,dt + C_1,\ \int -gt + v_0\sin\alpha\,dt + C_2 \right\rangle$$

$$= \left\langle v_0\cos\alpha\, t + C_1,\ \frac{-gt^2}{2} + v_0\sin\alpha\, t + C_2 \right\rangle$$

Finally $\vec{r}(0) = \langle x_0, y_0 \rangle$

$$\Rightarrow \langle C_1, C_2 \rangle = \langle x_0, y_0 \rangle$$

$$\Rightarrow \vec{r}(t) = \left\langle v_0\cos\alpha\, t + x_0,\ \frac{-gt^2}{2} + v_0\sin\alpha\, t + y_0 \right\rangle \ //$$

Thm. (Differentiation of vector-valued function)

(i) $\frac{d}{dt}\left(\vec{u}(t) \pm \vec{v}(t)\right) = \frac{d\vec{u}(t)}{dt} \pm \frac{d\vec{v}(t)}{dt}$

(ii) $\frac{d}{dt}\, c\,\vec{u}(t) = c\,\frac{d\vec{u}(t)}{dt}$ for any $c \in \mathbb{R}$

(iii) $\frac{d}{dt}\, h(t)\,\vec{u}(t) = h'(t)\,\vec{u}(t) + h(t)\,\vec{u}'(t)$ where $h(t)$ is any scalar or real-valued function.

(iv) $\frac{d}{dt}\left(\vec{u}(t)\cdot\vec{v}(t)\right) = \vec{u}'(t)\cdot\vec{v}(t) + \vec{u}(t)\,\vec{v}'(t)$

(v) $\frac{d}{dt}\left(\vec{u}(t)\times\vec{v}(t)\right) = \vec{u}'(t)\times\vec{v}(t) + \vec{u}(t)\times\vec{v}'(t)$

Pf: Excercise, all could be proved component-wise.

Date

Parametrized Curve in space

Given $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$, $a \le t \le b$ as the trajectory of an moving object in space. We could also visualize it as a space curve with the x, y and z co-ordinates of the points on the curve being parametrized by

$$\begin{cases} x = f(t) \\ y = g(t) \\ z = h(t) \end{cases} \quad a \le t \le b$$

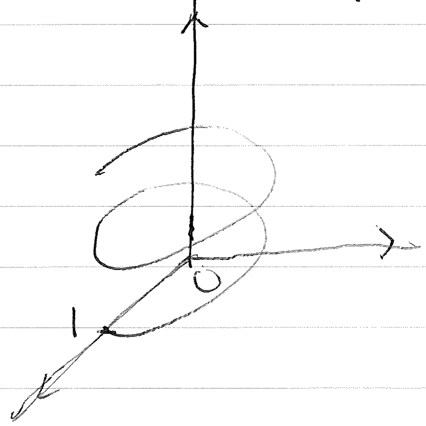
using t as a parameter. Then the length of the curve is given by

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$$

$$= \int_a^b \sqrt{f'(t)^2 + g'(t)^2 + h'(t)^2}\, dt$$

Ex Find the length of two loops of the helix

$$\begin{cases} x = \cos t \\ y = \sin t \\ z = t \end{cases}$$



2 loops would correspond to $0 \le t \le 4\pi$

$$\therefore L = \int_0^{4\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + 1}\, dt = \int_0^{4\pi} \sqrt{\sin^2 t + \cos^2 t + 1}\, dt = \int_0^{4\pi} \sqrt{2}\, dt$$

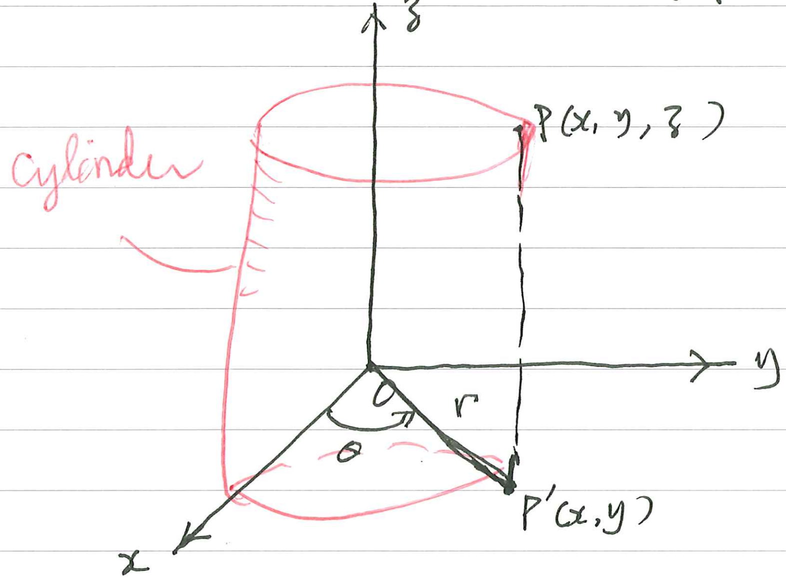
$$= \sqrt{2}\left[t\right]_0^{4\pi} = 4\pi\sqrt{2}$$

Date

Cylindrical and Spherical Co-ordinates

Cylindrical Co-ordinates

It is a combination of the polar co-ordinates (r, θ) in the x-y plane and the z-co-ordinate. Indeed, consider any point $P(x, y, z)$ in space, let P' be the projection of P onto the x-y plane.



$$\begin{cases} 0 \le r < \infty \\ 0 \le \theta < 2\pi \\ -\infty < z < \infty \end{cases}$$

We use (r, θ, z) to determine the position of P where we have replaced (x, y) by its polar co-ordinates of P' (its projection) while keeping the z-co-ordinates.

Remark: We call (r, θ, z) cylindrical co-ordinate because we are essentially visualizing the point P as residing on a vertical circular cylinder with radius r and with the z-axis as its central axis

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$$

$\leftarrow$ Conversion equations from $(r, \theta, z) \to (x, y, z)$

$$\begin{cases} r = \sqrt{x^2+y^2} \\ \cos\theta = \dfrac{x}{\sqrt{x^2+y^2}}, \quad \sin\theta = \dfrac{y}{\sqrt{x^2+y^2}} \\ z = z \end{cases}$$

$\leftarrow$ where θ is being determined uniquely by these formulas

These are conversion equations from $(x, y, z) \to (r, \theta, z)$.

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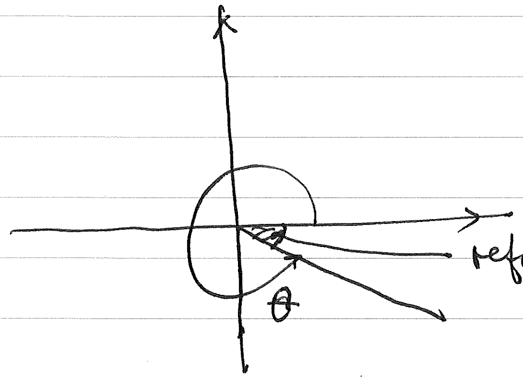
Ex Convert $(2, -2\sqrt{3}, 7) \longrightarrow (r, \theta, z)$

$z = 7$, $r = \sqrt{2^2 + (-2\sqrt{3})^2} = \sqrt{16} = 4$

θ is determined by $\cos\theta = \frac{2}{4} = \frac{1}{2}$, $\sin\theta = \frac{-2\sqrt{3}}{4} = \frac{-\sqrt{3}}{2}$

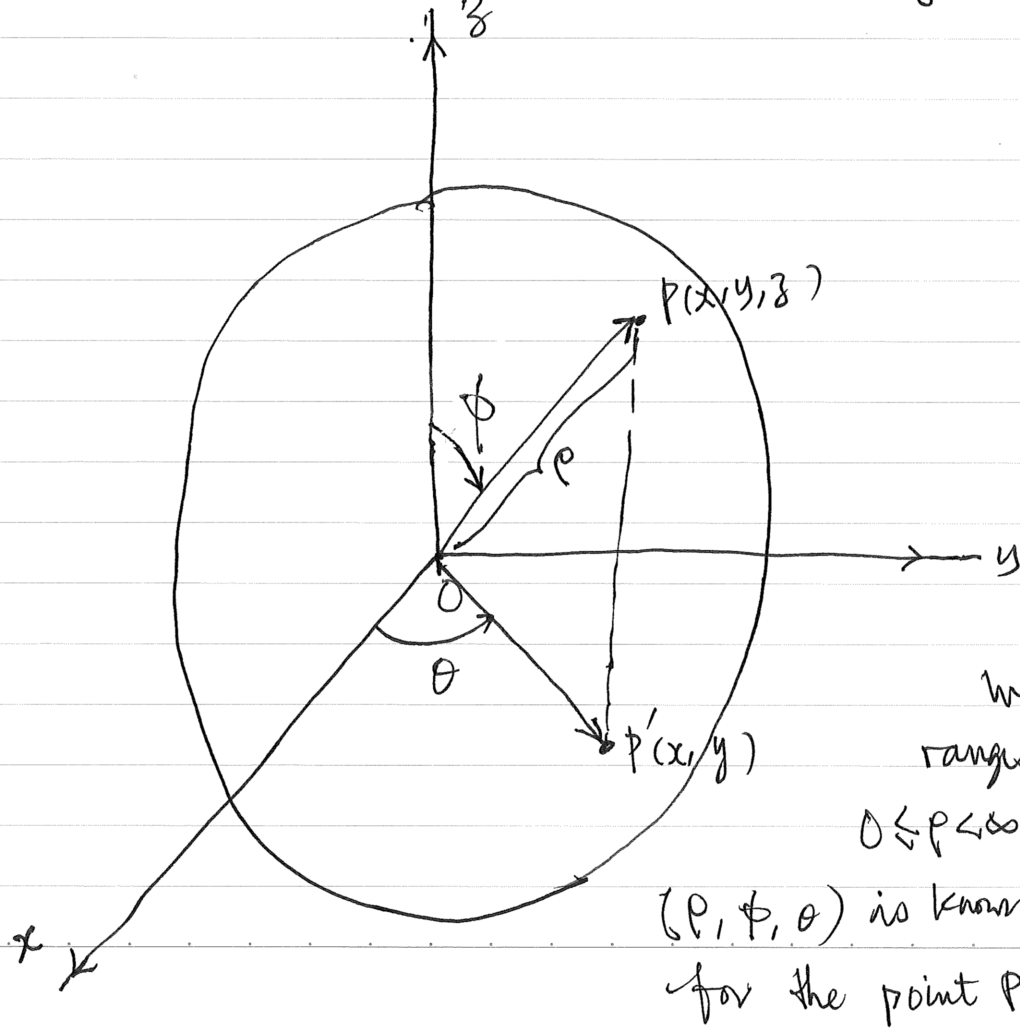


reference ∡ of $\theta = \pi/3$ $\therefore \theta = 2\pi - \pi/3 = 5\pi/3$

$\therefore (4, 5\pi/3, 7)$ is $(2, -2\sqrt{3}, 7)$ in cylindrical co-ordinates.

Spherical co-ordinates

In this case we visualize a point $P(x, y, z)$ in space as residing on a sphere with radius ρ centered at the original of the co-ordinate system.



$\rho = |OP|$ = distance of P from the origin

ϕ = angle that OP makes with the positive z axis

θ = the angle that OP′ makes with the positive x-axis as before

We could impose the following ranges on ρ, ϕ and θ

$0 \le \rho < \infty$, $0 \le \phi \le \pi$, $0 \le \theta < 2\pi$

(ρ, ϕ, θ) is known as the spherical co-ordinates for the point P.

From $(\rho, \phi, \theta) \longrightarrow (x, y, z)$
$$\begin{cases} x = \rho \sin\phi \cos\theta \\ y = \rho \sin\phi \sin\theta \\ z = \rho \cos\phi \end{cases}$$

From $(x, y, z) \longrightarrow (\rho, \phi, \theta)$
$$\begin{cases} \rho = \sqrt{x^2+y^2+z^2} \\ \phi = \cos^{-1}\left(\frac{z}{\rho}\right) \\ \theta \text{ being specified by } (x, y) \text{ as usual} \end{cases}$$

Ex Convert $P(-2, 4, -12)$ into its spherical co-ordinates (ρ, ϕ, θ)

$\rho = \sqrt{4+16+144} = \sqrt{164} = 2\sqrt{41}$

$\phi = \cos^{-1}\left(\frac{-6}{\sqrt{41}}\right)$

As for θ, we have $\cos\theta = \frac{x}{\sqrt{x^2+y^2}} = \frac{-2}{\sqrt{20}} = \frac{-1}{\sqrt{5}}$, $\sin\theta = \frac{4}{\sqrt{20}} = \frac{2}{\sqrt{5}}$

As $(-2, 4)$ belongs to the 2nd quadrant, we take $\theta = \cos^{-1}\left(\frac{-1}{\sqrt{5}}\right)$

Thus $\left(2\sqrt{41}, \cos^{-1}\left(\frac{-6}{\sqrt{41}}\right), \cos^{-1}\left(\frac{-1}{\sqrt{5}}\right)\right)$ is its spherical co-ordinates.

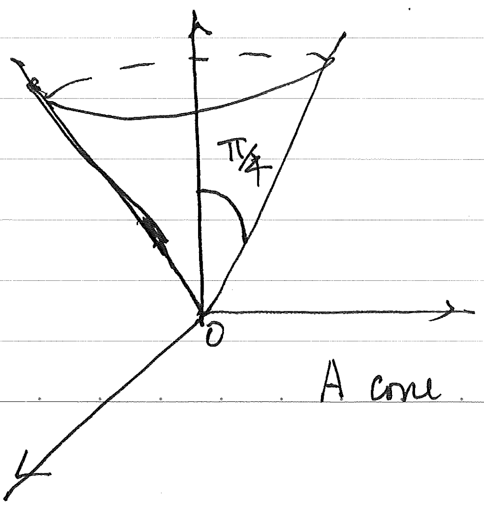
Ex. Describe and sketch the following surfaces whose equations are given in either cylindrical or spherical coordinates.

(i) $\phi = \pi/4$

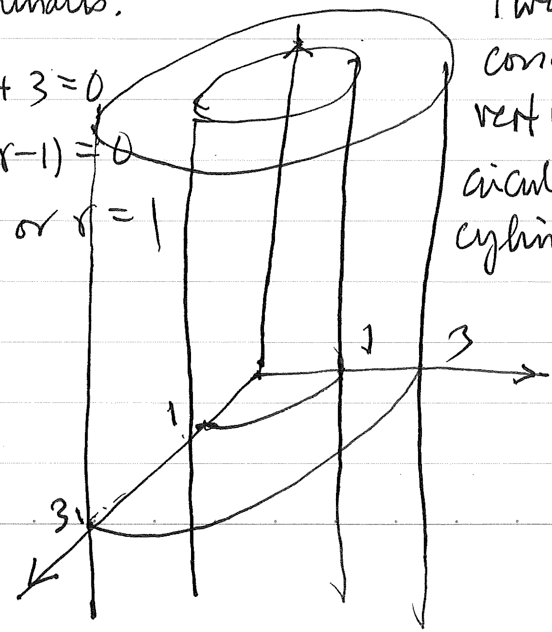
(ii) $r^2 - 4r + 3 = 0$

$\Rightarrow (r-3)(r-1) = 0$

$\Rightarrow r = 3$ or $r = 1$



A cone



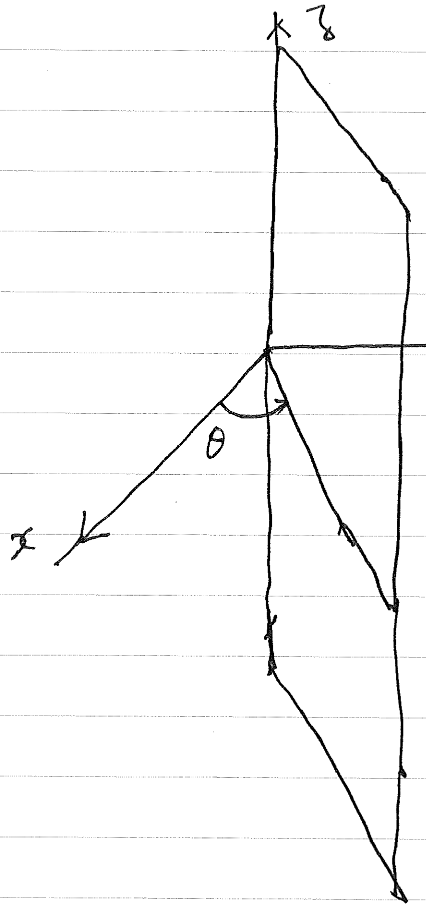
Two concentric vertical circular cylinder.

Date

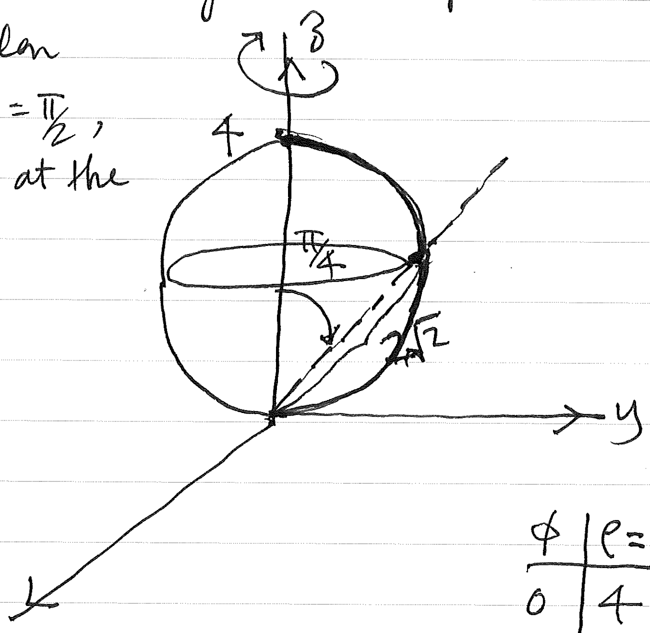
Note that in case (i) $\phi = \pi/4$, ρ and θ are not involved in the equation which means ρ and θ could be anything belonging to the ranges $0 \le \rho < \infty$ and $0 \le \theta < 2\pi$.

In case (ii) $r^2 - 4r + 3 = 0$, θ and z are not involved in the equation which means θ and z could take any values in their ranges $0 \le \theta < 2\pi$ and $-\infty < z < \infty$

As for case (iii) $\rho = 4\cos\phi$, θ is not involved in the equation, therefore could be anywhere in its range $[0, 2\pi)$, it suffices to consider any θ-plane (a vertical plane which makes an angle θ be the positive axis)



In particular we could take $\theta = \frac{\pi}{2}$, we'll be looking at the y-z plane



ϕ	$\rho = 4\cos\phi$
0	4
$\pi/4$	$2\sqrt{2}$
$\pi/2$	0

It yields a semi-circle on the y-z plane, as we let θ rotate a full range on $[0, 2\pi)$, we ended up getting a spherical surface centered at $(0, 0, 2)$ with radius 2. We could actually double check by converting its original equation back to its cartesian or rectangular form. Indeed,

$$\rho = 4\cos\phi \Leftrightarrow \rho^2 = 4\rho\cos\phi \Leftrightarrow x^2 + y^2 + z^2 = 4z$$

$$\Leftrightarrow x^2 + y^2 + (z-2)^2 = 4$$